Chained Permutations

Dylan Heuer

North Dakota State University

July 26, 2018
Three person chessboard
Three person chessboard
Three person chessboard
Three person chessboard - Rearranged
Two new families of chessboards

The board $B_{5,3}^-$

The board $B_{4,6}^\circ$
General enumerative result

**Theorem**

The number of ways to place \( m \) non-attacking rooks on board \( B \in \{ B_{n,k}^-, B_{n,k}^\circ \} \) is

\[
\sum_{(a_1, \ldots, a_k) \in \mathcal{C}_m(B)} \prod_{i=1}^{k} \binom{n - a_{i-1}}{a_i} (n)_{a_i}
\]

where \( a_0 \) is defined as:

\[
a_0 = \begin{cases} 
0 & \text{if } B = B_{n,k}^- \\
 a_k & \text{if } B = B_{n,k}^\circ .
\end{cases}
\]
Chained permutations

Maximum rook placement:

Permutation matrix form:
Chained permutations

Permutation matrix form:

One-line notation:

0200 – 3104 – 3000 – 3420 – 0004 – 1032–
Work towards an analog of Bruhat order

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- Thinking of a permutation in matrix form, we can think of an adjacent transposition as swapping adjacent rows.
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- Thinking of a permutation in matrix form, we can think of an adjacent transposition as swapping adjacent rows.
- There is a “natural” way to modify this in the case of chained permutations.
- We can perform a swap of adjacent rows on the $i$th matrix, while simultaneously performing a corresponding swap of adjacent columns on the $(i + 1)$st matrix.
Work towards an analog of Bruhat order

\[ \text{00012} - 00012 - 00123 - \]

\[ \downarrow s_{3,2} \]

\[ \text{00012} - 00102 - 00124 - \]
Work towards an analog of Bruhat order

- We can use these transpositions to generate a poset, just like with usual permutations.
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SageMath has been useful not only for its computational power, but also for its ability to visualize and work with graphs and posets.
Work towards an analog of Bruhat order

$k = 2, n = 3$, circular, fixed composition $(2,1)$
Work towards an analog of Bruhat order

Inversion number?

- It seems that there is a relatively “nice” analog of inversion number for chained permutations.
Work towards an analog of Bruhat order

\[ \text{00012} \rightarrow \text{00012} \rightarrow \text{00123} \rightarrow \]

\[ \downarrow_{s_3,1} \]

\[ \text{00102} \rightarrow \text{00012} \rightarrow \text{00123} \rightarrow \]
Work towards an analog of Bruhat order

\[
\begin{array}{c}
00012 \rightarrow 00012 \rightarrow 00123
\end{array}
\]

\[
\downarrow s_{1,1}
\]

\[
\begin{array}{c}
00012 \rightarrow 00021 \rightarrow 00123
\end{array}
\]
Work towards an analog of Bruhat order

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\]

\[s_2,1\]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{array}
\]

\[00012 \rightarrow 00013 \rightarrow 00123\]
Work towards an analog of Bruhat order

\[ 00012 - 00012 - 00123 - \]

\[ \downarrow s_{3,2} \]

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Inversion number?

- It seems that there is a relatively “nice” analog of inversion number for chained permutations.
- We can start with a chained permutation and algorithmically change it to the identity.
- In fact, it appears to be the case that using this analog of inversion number,

\[
\sum_{w \in P_{n,k}} q^{\text{inv}(w)} = \prod_{i=1}^{k} \left[ n - a_{i-1} \right]_{q}^{a_i} [n]_{a_i}
\]

(the \(q\)-analog of the counting formula), just as it is with usual permutations.
Thank you!